Deriving Range and Cross-range Distances in Spherical Coordinates

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Consider a missile or hypersonic glider being launched north from the point P_0 = (latitude, longitude) = $(\theta, \phi) = (0, 0)$. The range direction is along the great circle defined by the initial direction of motion—in this case the range direction is a line of longitude. If the missile's trajectory deviates laterally from that direction, you can also define its cross-range at any time.

Assume at some time the missile has reach the point $P_1 = (\theta_1, \phi_1)$. The cross-range is measured along a great circle that passes through P_1 and is perpendicular to the great circle that lies along the range direction. Call Q the point where the range and cross-range great circles cross. Then the range is the ground distance along the great circle between P_0 and Q, and the cross-range is the ground distance along the great circle between P_1 and Q.

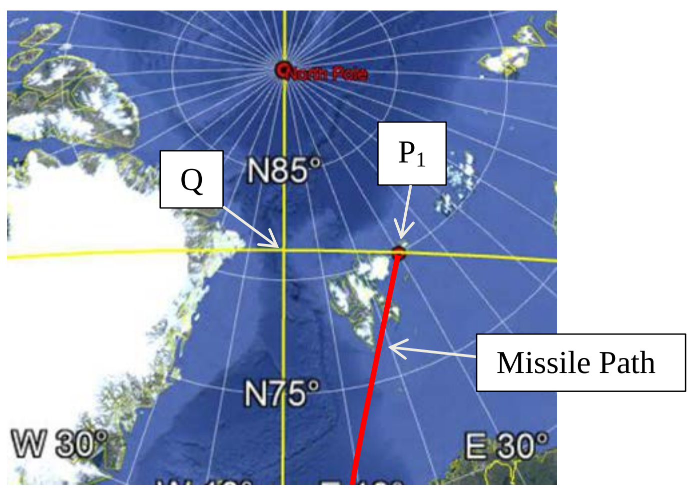

Figure 1: Po is off the bottom of the figure, lying on the yellow longitude line.

These distances can be calculated from the coordinates of P_1 using spherical geometry.

Calculating Cross-Range:

Consider a spherical triangle with sides made of segments of great circles as in Figure 2. Take *A* to be at the north pole, and angles *B* and *C* to be the located where a great circle crosses the circle of latitude θ_1, so that *C* is at the point P_1 in Figure 1.

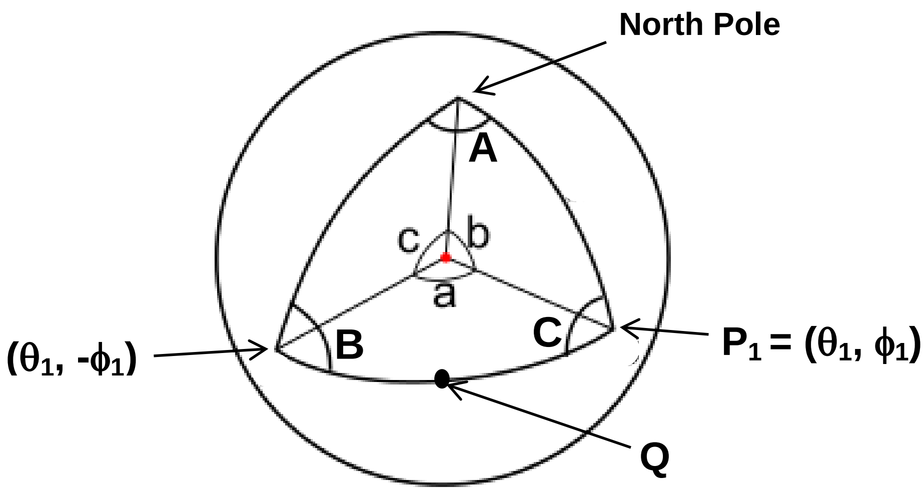

Figure 2: All the arcs in this figure are segments of great circles and the red dot marks the center of the sphere, so the straight lines are radii to points on the sphere. The letters *A*, *B*, *C*, *a*, *b*, *c* label the angles shown. The points labeled correspond to points in Figure 1.

Then angle $A = 2\phi_1$, and angles b and c are $90^0 - \theta_1$.

Then the spherical law of cosines:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad (1)$$

gives:

$$\cos a = \sin^2\theta_1 + \cos^2\theta_1 \cos(2\phi_1) \qquad (2)$$

and

$$Crossrange = \frac{a}{2} r_e \qquad (3)$$

Calculating Range:

The range angle corresponding to a missile at the point P_1 can be found in a similar way.

Consider the spherical triangle in Figure 3, which now represents the right half of the triangle in Figure 2; the point Q from Figure 1 is now at the left end of the lower arc rather than in the middle. The arc between the north pole and Q is part of the line of longitude = 0, which is the range direction. That arc bisects the angles *A* and *a* in Figure 2. Since the missile is assumed to be launched from the equator, the range angle is given by $90^0 - d$.

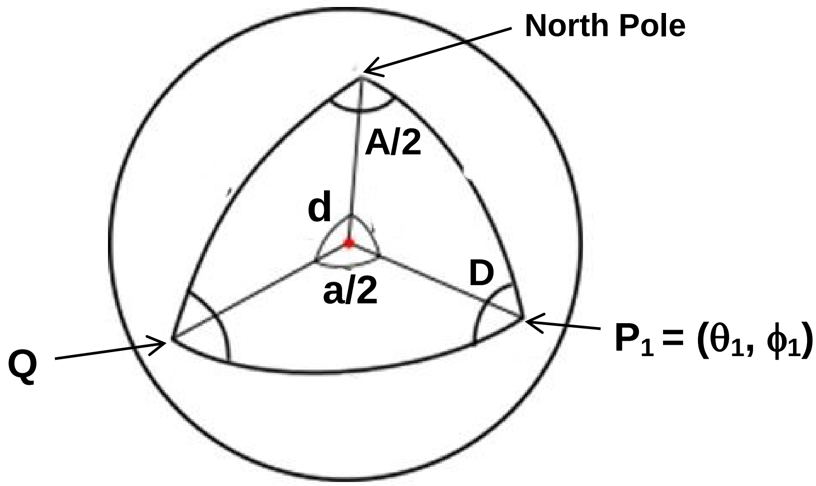

Figure 3: All the arcs in this figure are segments of great circles and the red dot marks the center of the sphere, so the straight lines are radii to points on the sphere. The letters *A/2*, *D*, *a/2*, and *d* label the angles shown. The points labeled correspond to points in Figure 1, and *A/2* and *a/2* are one-half the angles *A* and *a* in Figure 2. The angle *D* equals the angle *C* in Figure 2, but the angle *d* does not equal angle *c*.

Using the spherical law of sines:

$$\frac{\sin\frac{A}{2}}{\sin\frac{a}{2}} = \frac{\sin D}{\sin d} \qquad (4)$$

So that

$$\sin d = \sin D\,\frac{\sin\frac{a}{2}}{\sin\frac{A}{2}} \qquad (5)$$

From Figure 2:

$$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c} = \frac{\sin D}{\sin c} \qquad (6)$$

where the last equality comes from the fact that angles *C* and *D* are the same in Figures 2 and 3.

Combining these and using $A = 2\phi_1$ and $c = 90^0 - \theta_1$ gives:

$$\sin d = \left(\frac{\sin\frac{a}{2}}{\sin a}\right)\left(\frac{\sin 2\phi_1}{\sin\phi_1}\right)\cos\theta_1 = \frac{\cos\phi_1\cos\theta_1}{\cos\frac{a}{2}} \qquad (7)$$

or

$$\sin d = \frac{\cos\phi_1\cos\theta_1}{\sqrt{\frac{1+\cos a}{2}}} \qquad (8)$$

where *cos a* is given by Equation 2. The range is then given by:

$$Range = \left(\frac{\pi}{2} - d\right) r_e \qquad (9)$$

So for a missile at point (θ_1, ϕ_1), the range and cross-range distances are given by Equations (3) and (9).

Letting $\varepsilon = a/2$, so that ε is the cross-range angle, these equations can be written in an alternate form as:

$$cos^2\,\varepsilon = \left(1 + sin^2\theta_1 + cos^2\theta_1 \cos(2\phi_1)\right)/2 \qquad (10)$$

$$crossrange = \varepsilon r_e \qquad (11)$$

$$\text{range} = r_e\, cos^{-1}\left(\frac{\cos\theta_1 \cos\phi_1}{\cos\varepsilon}\right) \qquad (12)$$

To check these equations in one specific case, consider the case $(\theta_1,\phi_1) = (85^0, 90^0)$ (see Figure 4).

Notice that for a missile on this trajectory, the latitude angle θ never gets larger than 85^0, which makes clear that it cannot be treated as the range angle. In this case:

Eqs. 2 and 3 give:	*cos a* = 0.984908 → $a/2 = \varepsilon = 5^0$ Cross-range = 0.08727 r_e = 556 km
Eqs. 8 and 9 give:	*sin d* = 0 → $d = 0^0$ → range angle = 90^0 Range = 1.571 r_e = 10,006 km

Eqs. (10) – (12) give the same results.

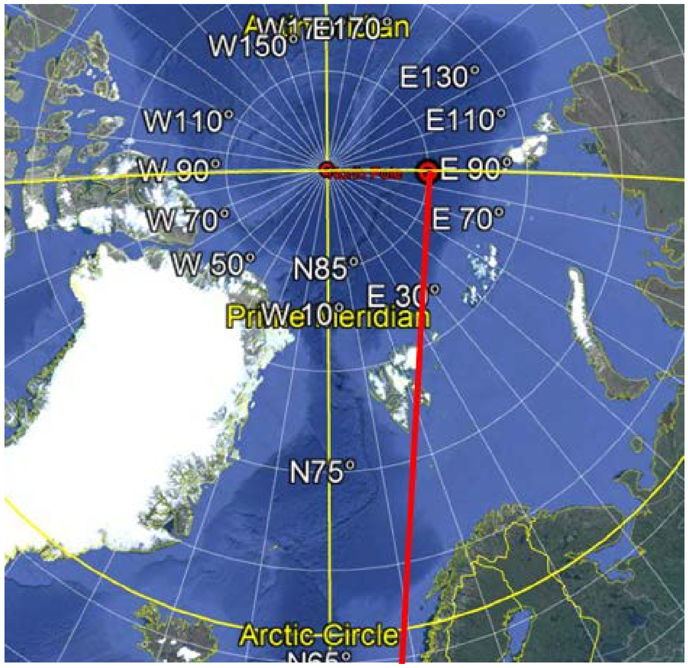

Figure 4. This figure shows the final part of a trajectory (in red) starting at (0, 0) and lying past the north pole. In this case it is clear that the range angle is 90^0 and the cross-range angle is 5^0.